We have the system

$$\left\{\begin{array}{rcrcrcl} x_1 & + & 2x_2 & + & x_3 & = & 0 \\ 2x_1 & + & 5x_2 & & & = & 1 \\ & & x_2 & + & 2x_3 & = & 9 \end{array}\right. \xRightarrow{\text{want to translate to}} \left\{\begin{array}{rcl} x_1 & & = ? \\ & x_2 & = ? \\ & & x_3 = ? \end{array}\right.$$

In matrix notation, we aim to obtain

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 2 & 5 & 0 & 1 \\ 0 & 1 & 2 & 9 \end{array}\right] \xrightarrow{\text{somehow}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & ? \\ 0 & 1 & 0 & ? \\ 0 & 0 & 1 & ? \end{array}\right]$$

Want to eliminate variables in the equations of the system. What are things we can do (without changing the solution set)?

- Interchange 2 equations $\longrightarrow$ Interchange 2 rows

- Multiply an equation by a constant $\longrightarrow$ Multiply a row by a constant

- Add/subtract a constant multiple of one row to another $\longrightarrow$ Add/subtract a constant multiple of one row to another.

Obviously we can switch equations in the system, add or subtract them, or multiply by constants without changing anything. This translates to <u>row operations</u> on the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 2 & 5 & 0 & 1 \\ 0 & 1 & 2 & 9 \end{array}\right] \xrightarrow{-2R_1 + R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 1 & 2 & 9 \end{array}\right] \longrightarrow$$

$$\xrightarrow{-R_2 + R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 4 & 8 \end{array}\right] \xrightarrow{\frac{1}{4}R_3} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$4x_3 = 8 \Rightarrow x_3 = 2$

$$\xrightarrow[-R_3 + R_1 \to R_1]{2R_3 + R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 2 & 0 & -2 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 2 \end{array}\right] \xrightarrow{-2R_2 + R_1 \to R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -12 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

which translates to

$$\begin{cases} x_1 = -12 \\ x_2 = 5 \\ x_3 = 2 \end{cases}$$

as the unique solution.

Elementary Row Operations

1) Replace one row by the sum of itself and a scalar multiple of another row

2) Interchange two rows

3) Multiply all entries in a row by a nonzero constant

Example

Is the system
$$\begin{cases} 2x_1 + 6x_2 + 4x_3 = 2 \\ 3x_1 + 4x_2 + 2x_3 = 2 \\ 5x_2 + 4x_3 = 8 \end{cases}$$
consistent?

$$\left[\begin{array}{ccc|c} 2 & 6 & 4 & 2 \\ 3 & 4 & 2 & 2 \\ 0 & 5 & 4 & 8 \end{array}\right] \xrightarrow{-\frac{3}{2}R_1 + R_2 \to R_2} \left[\begin{array}{ccc|c} 2 & 6 & 4 & 2 \\ 0 & -5 & -4 & -1 \\ 0 & 5 & 4 & 8 \end{array}\right]$$

$$\xrightarrow{R_2 + R_3 \to R_3} \left[\begin{array}{ccc|c} 2 & 6 & 4 & 1 \\ 0 & -5 & -4 & -1 \\ 0 & 0 & 0 & 7 \end{array}\right]$$

$$0x_1 + 0x_2 + 0x_3 = 7$$

Clearly $0 \neq 7$

so there are no solutions so the system is <u>inconsistent</u>.

§1.2 Row Reduction and Echelon Forms

Defn's

- A <u>nonzero row/column</u> is a row/column containing at least one nonzero entry
- The <u>leading entry</u> of a nonzero row is the leftmost nonzero entry

Echelon Forms

We say a matrix is in <u>Echelon form</u> if it satisfies the following

1) All nonzero rows are above any rows of zeroes

2) The leading entry of a row is in a column to the right of the leading entry of the row above it.

3) All entries in a column below a leading entry are zeroes

For example

zeroes below leading entries

$$\left[\begin{array}{ccc|c} 2 & 3 & 4 & 8 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \text{and} \quad \left[\begin{array}{cccc|c} 1 & 4 & 2 & -5 & 0 \\ 0 & 0 & 1 & 8 & 9 \\ 0 & 0 & 0 & 2 & 0 \end{array}\right]$$

Row of zeroes at bottom

Doesn't matter if augmented matrix or not.

Basically a matrix is in Echelon form if the zero entries make a "staircase" down the rows.

$$\begin{bmatrix} 0 & \# & \# & \# & \# & \# & \# & \# \\ 0 & 0 & 0 & \# & \# & \# & \# & \# \\ 0 & 0 & 0 & 0 & \# & \# & \# & \# \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \# \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

where #'s are any number and the circled ones are nonzero.

We say a matrix is in <u>reduced echelon form</u> if it is in echelon form, i.e. satisfies properties 1) - 3), and additionally

4) The leading entry in each nonzero row is 1.

5) Each leading 1 is the only nonzero entry in its column

If we take the matrix above, it would be in reduced echelon form if it looked like